MIT OCW GR 8.962 PSET 10

1. The schwarzchild line element is given by:

$$ds^2 = -\left(1 - \frac{2GM}{r}\right)dt^2 + \frac{dr^2}{(1-2GM/r)} + r^2 d\Omega^2$$

- perform the transformation:

$$r = \bar{r}\left(1 + GM/2\bar{r}\right)^2$$

$$\rightarrow dr = d\bar{r}\left(1 + GM/2\bar{r}\right)^2 + 2\bar{r}\left(1 + GM/2\bar{r}\right)\left(-GM/2\bar{r}^2\right) \cdot d\bar{r}$$

$$\rightarrow dr = d\bar{r}\left[\left(1 + \frac{GM}{2\bar{r}}\right)^2 - \left(\frac{GM}{\bar{r}}\right)\left(1 + \frac{GM}{2\bar{r}}\right)\right]$$

$$\rightarrow dr^2 = d\bar{r}^2\left[\left(1 + \frac{GM}{2\bar{r}}\right)^4 + \left(\frac{GM}{\bar{r}}\right)^2\left(1 + \frac{GM}{2\bar{r}}\right)^2 - 2\left(\frac{GM}{\bar{r}}\right)\left(1 + \frac{GM}{2\bar{r}}\right)^3\right]$$

$$= d\bar{r}^2\left(1 + \frac{GM}{2\bar{r}}\right)^2\left[\left(1 + \frac{GM}{2\bar{r}}\right)^2 + \left(\frac{GM}{\bar{r}}\right)^2 - 2\left(\frac{GM}{\bar{r}}\right)\left(1 + \frac{GM}{2\bar{r}}\right)\right]$$

$$= d\bar{r}^2\left(1 + \frac{GM}{2\bar{r}}\right)^2\left[1 + \frac{GM}{\bar{r}} + \left(\frac{GM}{2\bar{r}}\right)^2 + \cancel{\left(\frac{GM}{\bar{r}}\right)^2} - \frac{2GM}{\bar{r}} - \cancel{\left(\frac{GM}{\bar{r}}\right)^2}\right]$$

$$= d\bar{r}^2\left(1 + GM/2\bar{r}\right)^2\left(1 - GM/2\bar{r}\right)^2 = dr^2$$

· Now plug these definitions back into the line element:

$$ds^2 = -\left(1 - \frac{2GM}{\bar{r}(1+\frac{GM}{2\bar{r}})^2}\right)dt^2 + \frac{(1+\frac{GM}{2\bar{r}})^2(1-\frac{GM}{2\bar{r}})^2 d\bar{r}^2}{\left(1 - \frac{2GM}{\bar{r}(1+\frac{GM}{2\bar{r}})^2}\right)}$$

$$+ \bar{r}^2\left(1+\frac{GM}{2\bar{r}}\right)^4 d\Omega^2$$

· Define $\star \equiv \left(1 - \frac{2GM}{\bar{r}(1+GM/2\bar{r})^2}\right)$

$$= \frac{\bar{r}(1+GM/2\bar{r})^2 - 2GM}{\bar{r}(1+GM/2\bar{r})^2}$$

$$= \frac{\cancel{\bar{r}}(1-GM/2\bar{r})^2}{\cancel{\bar{r}}(1+GM/2\bar{r})^2}$$

$$\rightarrow ds^2 = \frac{-(1-GM/2\bar{r})^2}{(1+GM/2\bar{r})^2}dt^2 + (1+GM/2\bar{r})^4 d\bar{r}^2$$

$$+ \bar{r}^2(1+GM/2\bar{r})^4 d\Omega^2$$

$\rightarrow$

$$\rightarrow ds^2 = g_{tt}(\bar{r})\,dt^2 + g(\bar{r})(d\bar{r}^2 + \bar{r}^2 d\Omega^2)$$

<u>where</u>:

$$g_{tt}(\bar{r}) \equiv \frac{-(1 - GM/2\bar{r})^2}{(1 + GM/2\bar{r})^2}$$

– and –

$$g(\bar{r}) \equiv \left(1 + \frac{GM}{2\bar{r}}\right)^4$$

which shows there is a fundamental isotropy in the spatial coordinates. Hence, these are called isotropic coordinates ✓✓

$\boxed{6}$. Take the limit $\bar{r} \gg GM$:

$$\lim_{\bar{r} \gg GM} g_{tt}(\bar{r}) \longrightarrow \frac{-1^2}{1^2} = -1$$

$$\lim_{\bar{r} \gg GM} g(\bar{r}) \longrightarrow 1$$

Implying the metric overall goes to $\rightsquigarrow$

$$\boxed{ds^2 \simeq -dt^2 + d\bar{r}^2 + \bar{r}^2 d\Omega^2}$$

↳ And this is just the metric of flat spacetime in spherical coordinates!!

$\boxed{2}$ Numerical construction of neutron star models in GR:

- A moderately accurate approximation to the EOS of the material which makes up a neutron star is given by the polytropic form:

$$P = K \rho_0^{\Gamma}$$

where P is pressure, ρ_0 is the <u>rest</u> matter density, and $\Gamma = 5/3$, and

$$K = \frac{3^{2/3} \pi^{4/3} \hbar^2}{5 m_n^{8/3}} \approx 5.38 \times 10^9 \ gm^{-2/3} \ cm^4 \ sec^{-2}$$

- In this problem we will numerically integrate the TOV equations of stellar structure to build models of neutron stars in GR ⟶

[redacted]

- Note that there is a difference between rest mass density + relativistic energy density ρ_0 v.s. ρ :

$$\rho = \rho_0 + \frac{P}{\Gamma - 1} = \rho_0 + \frac{K\rho_0^{\Gamma}}{\Gamma - 1}$$

- We also have:

$$\frac{dm}{dr} = 4\pi \rho r^2 \quad ; \quad \frac{dP}{dr} = \frac{-(\rho + P)(m + 4\pi r^3 P)}{r(r-2m)}$$

and we apply the boundary conditions:

$$m(r=0) = 0 \quad ; \quad P(r=0) \equiv P_c = P(\rho_{0,c}) = K\rho_{0,c}^{\Gamma}$$

- We integrate these equations until the pressure drops to zero $\underbrace{P(R_*)}_{=0}$ which defines the surface of the star ... The total mass of the star is then $m(R_*) \equiv M_*$

$\boxed{a}$. Numerical calculations of this kind work best when the system parameters are of order unity (computers don't like super small or super large floating point numbers). We will therefore use Geometrized units with $G = c = 1$ and set all our values to powers of kilometers... In the end we should find $M_* \sim 1$ km and $R_* \sim 10$ km as the appropriate orders of magnitude.

$\boxed{i}$. convert $[\rho] = \frac{1\,gm}{cm^3}$ to km^{-2} :

$G \approx 6.67 \times 10^{-8}\ cm^3 \cdot gm^{-1} \cdot sec^{-2}$

and $c \approx 3 \times 10^{10}\ cm \cdot sec^{-1}$

$$\rightarrow \left[\frac{\rho G}{c^2}\right] = cm^{-2} \quad \text{and} \quad \frac{1}{\cancel{cm^2}} \cdot \frac{10^{10}\ \cancel{cm^2}}{1\ km^2} \approx 10^{10}\ km^{-2}$$

$$\boxed{\text{So to convert } \rho \text{ to } km^{-2}, \text{ multiply by: } \frac{10^{10}\, G}{c^2} \approx \frac{(10^{10})(6.67 \times 10^{-8})}{(9 \times 10^{20})} \approx 7.41 \times 10^{-19}}$$

(ii) · Convert $P = 1\ gm \cdot cm^{-1} \cdot sec^{-2}$ to km^{-2}:

$\rightarrow \left[\frac{PG}{c^4}\right] = cm^{-2}$

· So to convert P to km^{-2} multiply by:

$$\boxed{\frac{10^{10} G}{c^4} \approx \frac{(10^{10})(6.67 \times 10^{-8})}{81 \times 10^{40}} \approx 8.23 \times 10^{-40}}$$

iii · Convert $K = 1\ gm^{-2/3} \cdot cm^4 \cdot sec^{-2}$ to $km^{4/3}$

$$\left[\frac{K}{(cG)^{2/3}}\right] = cm^{4/3}$$

· So to convert K to $km^{4/3}$ multiply by:

$$\left(10^{-20/3}\right)(cG)^{-2/3} \approx 10^{-20/3}\, 3^{-2/3}\, 6.67^{-2/3}\, 10^{-20/3}\, 10^{\frac{16}{3}}$$

$$\approx 1.36 \times 10^{-9}$$

· Given $K \approx 5.38 \times 10^9$ that means in geometrized units $\boxed{K \approx 7.299\ km^{4/3}}$

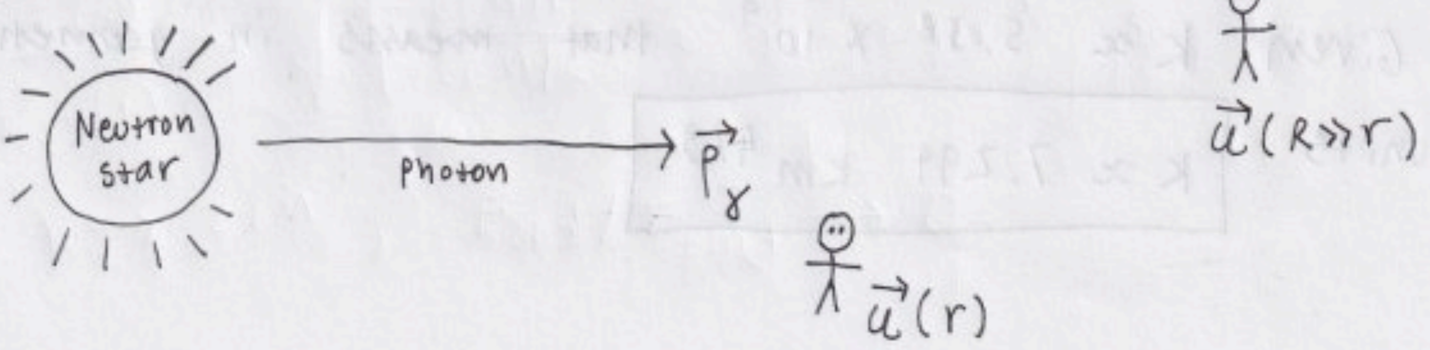
b. Pick a central density $\rho_{0,c} \approx 10^{15}\ gm/cm^3$ and perform the numerical integration to find R_* and M_* here: see the attached / accompanying Jupyter Notebook code + documentation ...

$\boxed{\text{I found that } M_* \approx 1.0 \text{ and } R_* \approx 12.02}$

c. If a photon is emitted radially with energy E_{em} from the surface of this star, what is the energy E_{obs} with which this photon is observed by distant observers at $r \to \infty$? Using these energies compute the redshift

$$z_{surf} = \frac{E_{obs} - E_{em}}{E_{obs}}$$

Picture



- We consider 2 observers that are stationary observing the emitted photon at r and $R \gg r$.
- The photon's 4-mom. observed by either observer is $\vec{p}_\gamma = \hbar\omega(-1, \hat{k})$ where $\hat{k}$ is the unit vector in the direction of propagation. The key thing to note is that p_t = constant since the Schwarzchild metric $ds^2 = \ldots$ is time-independent.
- We also remember the fact that $E_{\vec{u}}$ the energy of an object with 4-mom. $\vec{p}$ as seen by an observer with 4-vel. $\vec{u}$ is:

$$\boxed{E_{\vec{u}} = -\vec{p} \cdot \vec{u}}$$

- Finally, we can find $\vec{u}(r)$ for each observer:

$$-1 = \vec{u} \cdot \vec{u} = g_{\alpha\beta} u^\alpha u^\beta = g_{tt}(u^t)^2 + g_{ij} u^i u^j$$

but $u^i = u^j = 0$ since these observers are stationary:

$$\rightarrow -1 = g_{tt}(u^t)^2 = -\left(1 - \frac{2GM}{r}\right)(u^t)^2$$

$$\longrightarrow \quad u^t = (1 - 2GM/r)^{-1/2}$$

$$\longrightarrow \quad z_{surf} \equiv \frac{E_{obs}(R) - E_{emit}(r)}{E_{obs}(R)}$$

$$= \frac{P_t u^t(R) - P_t u^t(r)}{P_t u^t(R)}$$ and cancel P_t terms since constant!

$$= \frac{u^t(R) - u^t(r)}{u^t(R)} = 1 - \sqrt{\frac{1 - 2GM/R}{1 - 2GM/r}}$$

and as $R \gg r$ we get the limit:

$$\boxed{z_{surf} \approx 1 - (1 - 2GM/r)^{-1/2}}$$

radius of neutron star

- It is more insightful to consider the ratio:

$$\frac{E_{obs}(R)}{E_{emit}(r)} = \sqrt{\frac{1 - 2GM/r}{1 - 2GM/R}} \approx \sqrt{1 - 2GM/r}$$ so as

$$r \rightarrow 2GM \; ; \; E_{obs}/E_{emit} \rightarrow 0$$

· and no light emitted from $r = 26M$ or below can be seen by a distant observer. This is a boundary of "infinite redshift"... ✓

$\boxed{d}$ · As described in lecture, M_* is not what one would get by integrating all the fluid density elements over the proper volume of the star's interior. Let's define M_p as such the mass we would obtain from this integration. Re-integrate the TOV equations with:

$$\frac{dm_p}{dr} = 4\pi \rho r^2 \sqrt{g_{rr}} = 4\pi \rho r^2 \left(1 - \frac{2m(r)}{r}\right)^{-1/2}$$

· You can again check the Jupyter Notebook I attached. I found that $\boxed{M_p \approx 1.08}$

$\boxed{e}$ · The gravitational binding energy of the star Δ is then defined as: $\Delta \equiv \frac{M_p - M_*}{M_*}$

· I found $\boxed{\Delta \approx 0.08}$ which matches the same order of magnitude for the binding energy of a neutron star quoted else-where on the internet... ✓

3 Stability of a TOV Star

- By computing a range of TOV models, we can assess whether a star is stable against radial perturbations. Stable stars satisfy:

$dM/d\rho_c > 0 \rightarrow$ you push inwards on the star + it maintains its shape pushing back out doing work + increasing its energy / mass

- Unstable stars satisfy:

$dM/d\rho_c < 0 \rightarrow$ you push inwards on the star + it collapses into a black hole with a runaway reaction.

a . Use your integration code to find M_* for $\rho_{0,c} \in \{10^{14}, 10^{15}, 10^{16}, 10^{17}, 10^{18}\}$ and plot the results. Find the region where $dM/d\rho_c$ changes sign indicating a semi-stable region:

- This is plotted in the attached Jupyter NB. For this level of granularity, the semi-stable region occurs around $\boxed{\rho_{0,c} = 10^{16}}$

$\boxed{6}$ · Zoom in on this region + estimate M_*:

· This is done in the attached Jupyter NB. I found that $M_{*,\,\text{semi-stable}} \approx 1.17$ km in geometrized units. Now convert this to kg + solar masses:

$$M_* \approx (1.17\ \text{km})\,\frac{c^2}{G} = \frac{(1.17\ \text{m})(9 \times 10^{16}\ \text{m}^2/\text{s}^2)(10^3)}{(6.67 \times 10^{-11}\ \text{kg}^{-1} \cdot \text{m}^3 \cdot \text{s}^{-2})}$$

$$\boxed{M_* \approx 1.58 \times 10^{30}\ \text{kg}}$$

$$M_{sun} \approx 1.989 \times 10^{30}\ \text{kg}$$

$$\rightarrow \boxed{M_* \propto 0.79\ M_{sun}}$$

· Most neutron stars are about $2 \times M_{sun}$ so this is slightly below what we should expect but the right order of magnitude